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On the Bernstein–Bézier form of Jacobi polynomials on a simplex

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Abstract

Here, we give a simple proof of a new representation for orthogonal polynomials over triangular domains which overcomes the need to make symmetry destroying choices to obtain an orthogonal basis for polynomials of fixed degree by employing redundancy. A formula valid for simplices with Jacobi weights is given, and we exhibit its symmetries by using the Bernstein–Bézier form. From it, we obtain the matrix representing the orthogonal projection onto the space of orthogonal polynomials of fixed degree with respect to the Bernstein basis. The entries of this projection matrix are given explicitly by a multivariate analogue of the ${}_3F_2$ hypergeometric function. Along the way we show that a polynomial is a Jacobi polynomial if and only if its Bernstein basis coefficients are a Hahn polynomial. We then discuss the application of these results to surface smoothing problems under linear constraints.

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1. Introduction

This paper considers orthogonal polynomials over a triangular (or simplicial) domain, with a mind to extending least-squares approximation methods to the multivariate setting (see the

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discussion in [10]). In the univariate case, these are given by the inner product

$$\langle f, g\rangle_\nu := \frac{\Gamma(\nu_0+\nu_1)}{\Gamma(\nu_0)\Gamma(\nu_1)} \int_0^1 f(x)g(x)\,(1-x)^{\nu_0-1}x^{\nu_1-1}\,dx, \qquad \nu_0,\nu_1>0$$

with those for $\nu_j = 1$ (*Legendre polynomials*) and $\nu_j = \frac{1}{2}$ (*Chebyshev polynomials*), i.e.,

$$\langle f, g\rangle_{(1,1)} = \int_0^1 f(x)g(x)\,dx, \quad \langle f, g\rangle_{(\frac{1}{2},\frac{1}{2})} = \frac{1}{\pi}\int_0^1 f(x)g(x)\frac{dx}{\sqrt{1-x^2}},$$

the most relevant to least-squares methods. The linear polynomials $\xi_0(x) := 1-x$ and $\xi_1(x) := x$ above are the barycentric coordinates of the interval (1-simplex) $T = [0,1]$.

For polynomials of d variables the interval $[0,1]$ is replaced by a d-simplex T (a triangle for $d=2$) with barycentric coordinates $\xi = (\xi_0, \xi_1, \ldots, \xi_d)$, to obtain the inner product

$$\langle f, g\rangle_\nu := \frac{\Gamma(|\nu|)}{\Gamma(\nu)} \frac{1}{d!\,\mathrm{vol}_d(T)} \int_T fg\xi^{\nu-1}, \quad \nu \in \mathbb{R}^{d+1}, \ \nu_j > 0. \tag{1.1}$$

Let $\Pi_s(\mathbb{R}^d)$ be the space of polynomials of degree $\leqslant s$ on $\mathbb{R}^d$. Then $\mathcal{P}_s^\nu$ the *Jacobi polynomials* of degree s is defined to be the space of polynomials f of degree s with

$$\langle f, g\rangle_\nu = 0 \quad \forall g \in \Pi_{s-1}(\mathbb{R}^d).$$

In the univariate case ($d = 1$) this space is one dimensional, spanned by the orthogonal projection of any polynomial of exact degree s onto it. The issues here are the choice of an appropriate normalisation to give a neat form of the three term recurrence, and expressing the orthogonal polynomial in terms of a $({}_2F_1)$ hypergeometric function.

In the bivariate (and multivariate) case $\dim(\mathcal{P}_s^\nu) = \binom{s+d-1}{d-1} > 1$, $d > 1$, $s > 0$, and so some orthogonal-type expansion must be developed for it (see, e.g., [7]). Let us consider the issues involved here in the concrete case of the bivariate ($d = 2$) quadratic ($s = 2$) Legendre ($\nu_j = 1$) polynomials on the standard triangle $T := \{(x, y) : x, y \geqslant 0, x + y \leqslant 1\}$, which has barycentric coordinates $x, y, 1-x-y$. A natural candidate for an orthogonal basis would be the orthogonal projection of the Bernstein basis

$$x^2, y^2, (1-x-y)^2, 2xy, 2x(1-x-y), 2y(1-x-y)$$

onto $\mathcal{Q}$ the space of quadratic Legendre polynomials, since these Legendre polynomials are invariant under the symmetries of the weight (the affine changes of variables which map the triangle T to itself). *But* there are *six* of these functions and $\mathcal{Q}$ has dimension *three*! What can one do? Appell [1] suggests taking a subset: those corresponding to $x^2, y^2, 2xy$ (those not involving one of the barycentric coordinates). These are not orthogonal to each other: but they are invariant under a subgroup of the symmetries of the weight—enough to develop general formula for them and the dual basis. Proriol [11] suggests giving up on having any symmetries and obtains an orthogonal basis explicitly. The cost is that the (recursive) formulae are very complicated, and so of limited utility for computations. Here, Prorial's polynomials are the orthogonal projections of $x^2 + y^2 + 2xy, x^2 - y^2, x^2 - y^2 - 4xy$ onto $\mathcal{Q}$.

In this paper, we advocate a new approach: to write $f \in \mathcal{Q}$ as a sum of its orthogonal projections onto all six functions. The resulting formula is what is called a *tight frame* expansion (see [3]). It has a simple form since, e.g., the orthogonal projections of $x^2, y^2, (1-x-y)^2$ are obtained from each other by applying a symmetry of the weight (interchanging barycentric coordinates).

We advocate that having a simple formula which reflects the symmetries of the weight and so allows stable calculations (see [9]) outweighs the cost of dealing with more functions than needed for a basis.

The paper is set out as follows. In the remainder of this section we give some basic definitions and facts. In Section 2, we investigate the Bernstein–Bézier coefficients of Jacobi polynomials. It turns out that these are Hahn polynomials and can be characterised by certain dependencies of the coefficients which can be expressed in terms of the adjoint of the degree elevation operator. In Section 3, we give a tight frame representation for the space of Jacobi polynomials which reflects the symmetries of the weight and discuss its use. The proof given is based on the fact that the Jacobi polynomials are eigenfunctions of the Bernstein–Durrmeyer operator. We conclude by presenting some additional consequences of the results including their application to surface smoothing problems.

Throughout $\xi = (\xi_0, \xi_1, \dots, \xi_d)$ will be the barycentric coordinates of a d-simplex $T \subset \mathbb{R}^d$ (the convex hull of $d+1$ affinely independent points in $\mathbb{R}^d$) with volume $\mathrm{vol}_d(T)$. We use standard multi-index notation, e.g., in (1.1) we have $\xi^{\nu-1} = \xi_0^{\nu_0-1}\xi_1^{\nu_1-1}\cdots\xi_d^{\nu_d-1}$, $|\nu| = \sum_j \nu_j$ and $\Gamma(\nu) = \prod_j \Gamma(\nu_j)$. The normalisation is chosen so that

$$\langle \xi^\alpha, \xi^\beta \rangle_\nu = \frac{(\nu)_{\alpha+\beta}}{(|\nu|)_{|\alpha|+|\beta|}}, \quad \alpha, \beta \in \mathbb{Z}_+^{d+1}, \tag{1.2}$$

where $(\nu)_\alpha := \prod_j (\nu_j)_{\alpha_j}$, with $(x)_n := x(x+1)\cdots(x+n-1)$ the Pochhammer symbol.

Each polynomial $f \in \Pi_n(\mathbb{R}^d)$ can be expressed in terms of the Bernstein basis

$$f = \sum_{|\alpha|=n} c_\alpha(f) B_\alpha = \sum_{|\alpha|=n} c_\alpha B_\alpha,$$

where the *Bernstein polynomials* of degree n are defined by

$$B_\alpha := \binom{|\alpha|}{\alpha} \xi^\alpha = \frac{|\alpha|!}{\alpha!}\,\xi^\alpha = \frac{n!}{\alpha!}\,\xi^\alpha, \quad |\alpha| = n, \ \ \alpha \in \mathbb{Z}_+^{d+1}.$$

This basis for $\Pi_n(\mathbb{R}^d)$ is ideally suited to representing polynomials on the simplex T (see [2]), and $c(f) = c^n(f) = (c_\alpha)_{|\alpha|=n}$ are referred to as the *Bernstein*(*–Bézier*) coefficients. By the multinomial theorem

$$f = \sum_{|\alpha|=n} c_\alpha B_\alpha = \sum_{|\alpha|=n} c_\alpha B_\alpha \left(\sum_{i=0}^d \xi_i \right)^j = \sum_{|\alpha|=n+j} (R^j c)_\alpha B_\alpha,$$

where the powers of the *degree raising operator* R are given by

$$(R^j c)_\alpha = \sum_{|\gamma|=j} \binom{j}{\gamma} \frac{(-\alpha)_\gamma}{(-|\alpha|)_j} c_{\alpha-\gamma}, \quad j = 0, 1, 2, \dots . \tag{1.3}$$

2. Jacobi polynomials and their Bernstein coefficients

Here, we think of functions $c : \alpha \mapsto c_\alpha$ defined on the simplex points

$$S_n := \{\alpha \in \mathbb{Z}_+^{d+1} : |\alpha| = n\},$$

such as the Bernstein coefficients of $f \in \Pi_n(\mathbb{R}^d)$, as polynomials of degree n in d-variables. This is done by identifying c with the unique polynomial of degree n on the d-dimensional affine subspace $\{x \in \mathbb{R}^{d+1} : x_0+x_1+\cdots+x_d = n\}$ which takes the value c_α at $\alpha \in S_n$. For example, by the multinomial theorem $f = 1 = \sum_{|\alpha|=n} B_\alpha$, and so 1 corresponds to the constant polynomial $c : \alpha \mapsto 1$. More generally we have:

Proposition 2.1. *Let $f = \sum_{|\alpha|=n} c_\alpha B_\alpha \in \Pi_n(\mathbb{R}^d)$ and $0 \leqslant s \leqslant n$. Then f has degree s if and only if $c : \alpha \mapsto c_\alpha$ is a polynomial of degree s.*

Proof. The polynomials B_β, $|\beta| = s$, are a basis for $\Pi_s(\mathbb{R}^d)$, and can be expressed

$$B_\beta = \sum_{|\alpha|=s} b_\alpha B_\alpha, \quad b_\alpha := \begin{cases} 1, & \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Let $j := n - s$. Then by (1.3) the Bernstein coefficients of $B_\beta = \sum_{|\alpha|=n} c_\alpha B_\alpha$ are

$$c_\alpha = (R^j b)_\alpha = \sum_{|\gamma|=j} \binom{j}{\gamma} \frac{(-\alpha)_\gamma}{(-|\alpha|)_j} b_{\alpha-\gamma} = \frac{j!}{(\alpha-\beta)!} \frac{(-\alpha)_{\alpha-\beta}}{(-n)_j} = \frac{j!(-1)^j}{(-n)_j} \frac{(-\alpha)_\beta}{(-\beta)_\beta}$$

since $(-\alpha)_{\alpha-\beta}(-\beta)_\beta = (-\alpha)_\beta(-\alpha+\beta)_{\alpha-\beta}$ and $(\alpha-\beta)! = (-1)^j(-\alpha+\beta)_{\alpha-\beta}$.

But $S_n \to \mathbb{R} : \alpha \mapsto (-\alpha)_\beta$, $|\beta| = s$ are a basis for the space of polynomials of degree s, and so we obtain the stated correspondence. □

We define an inner product on the space of polynomials $S_n \to \mathbb{R}$ of degree n by

$$\langle f, g \rangle_{v,n} := \sum_{|\alpha|=n} \frac{(v)_\alpha}{\alpha!} f(\alpha) g(\alpha). \tag{2.2}$$

The corresponding orthogonal polynomials of degree s are called *Hahn polynomials*, and we denote the space of them by $\mathcal{P}_s^{v,n}$, $0 \leqslant s \leqslant n$. We now show that a polynomial is a Jacobi polynomial if and only if its Bernstein coefficients are a Hahn polynomial.

Theorem 2.3. *Let $f = \sum_{|\alpha|=n} c_\alpha B_\alpha \in \Pi_n(\mathbb{R}^d)$ and $0 \leqslant s \leqslant n$. Then $f \in \mathcal{P}_s^v$ if and only if $c \in \mathcal{P}_s^{v,n}$, i.e., $c : \alpha \mapsto c_\alpha$ is a polynomial of degree s, and*

$$\langle c, p \rangle_{v,n} = \sum_{|\alpha|=n} \frac{(v)_\alpha}{\alpha!} c_\alpha p(\alpha) = 0,$$

for all polynomials $p : S_n \to \mathbb{R}$ of degree $< s$.

Proof. $f \in \mathcal{P}_s^v$ if and only if it is orthogonal to the spanning set $g = \xi^\beta$, $|\beta| < s$ for $\Pi_{s-1}(\mathbb{R}^d)$, i.e., by (1.2)

$$\langle f, g \rangle_v = \sum_{|\alpha|=n} c_\alpha \frac{n!}{\alpha!} \frac{(v)_{\alpha+\beta}}{(|v|)_{|\alpha|+|\beta|}} = \frac{n!}{(|v|)_{s+|\beta|}} \sum_{|\alpha|=n} \frac{(v)_\alpha}{\alpha!} c_\alpha (v+\alpha)_\beta = 0.$$

The result follows since $p : \alpha \mapsto (v+\alpha)_\beta$, $|\beta| < s$ span the space of polynomials of degree $< s$. □

This interpretation of the Hahn polynomials as the Bernstein coefficients of the Jacobi polynomials was found by Ciesielski [4] in the univariate case. By choosing specific p the condition on the Bernstein coefficients can be related to R_v^* the adjoint of the degre raising operator with respect to (2.2), which is defined by

$$\langle Rc, b\rangle_{v,n} = \langle c, R_v^* b\rangle_{v,n-1}, \quad c : S_{n-1} \to \mathbb{R}, \ \ b : S_n \to \mathbb{R}.$$

Choose $0 \leqslant j \leqslant n$. Then for $c : S_{n-j} \to \mathbb{R}$ and $b : S_n \to \mathbb{R}$ we calculate

$$\begin{aligned}\langle R^j c, b\rangle_{v,n} &= \sum_{|\alpha|=n} \frac{(v)_\alpha}{\alpha!} (R^j c)_\alpha b_\alpha = \sum_{|\alpha|=n} \frac{(v)_\alpha}{\alpha!} \sum_{|\gamma|=j} \binom{j}{\gamma} \frac{(-\alpha)_\gamma}{(-|\alpha|)_j} c_{\alpha-\gamma} b_\alpha \\ &= \sum_{|\beta|=n-j} \sum_{|\gamma|=j} \frac{(v)_{\beta+\gamma}}{(\beta+\gamma)!} \binom{j}{\gamma} \frac{(-\beta-\gamma)_\gamma}{(-n)_j} c_\beta b_{\beta+\gamma} \\ &= \sum_{|\beta|=n-j} \frac{(v)_\beta}{\beta!} c_\beta \sum_{|\gamma|=j} \beta! \frac{(v+\beta)_\gamma}{(\beta+\gamma)!} \binom{j}{\gamma} \frac{(-\beta-\gamma)_\gamma}{(-n)_j} b_{\beta+\gamma},\end{aligned}$$

and so the powers of the adjoint of R are given by

$$\begin{aligned}((R_v^*)^j b)_\beta &= \sum_{|\gamma|=j} \beta! \frac{(v+\beta)_\gamma}{(\beta+\gamma)!} \binom{j}{\gamma} \frac{(-\beta-\gamma)_\gamma}{(-n)_j} b_{\beta+\gamma} = \sum_{|\gamma|=j} (v+\beta)_\gamma \binom{j}{\gamma} \frac{(-1)^j}{(-n)_j} b_{\beta+\gamma} \\ &= \sum_{|\gamma|=j} \frac{(\beta+v)_\gamma}{(|\beta|+1)_j} \binom{j}{\gamma} b_{\beta+\gamma}. \qquad (2.4)\end{aligned}$$

Corollary 2.5. *Let* $f = \sum_{|\alpha|=n} c_\alpha B_\alpha \in \Pi_n(\mathbb{R}^d)$, $0 \leqslant s \leqslant n$. *Then* $f \in \mathcal{P}_s^v$ *if and only if*

$$(R_v^*)^{n-s+1} c = 0.$$

Proof. Take $p : \alpha \mapsto (-\alpha)_\beta$, $|\beta| = s-1$ which are a basis for the space of polynomials of degree $< s$. With $k := n - |\beta| = n - s + 1$, we calculate

$$\begin{aligned}\langle c, p\rangle_{v,n} &= \sum_{|\alpha|=n} \frac{(v)_\alpha}{\alpha!} c_\alpha (-\alpha)_\beta = \sum_{\substack{|\alpha|=n \\ \alpha \geqslant \beta}} \frac{(v)_\alpha}{\alpha!} c_\alpha (-\alpha)_\beta = \sum_{|\gamma|=k} \frac{(v)_{\beta+\gamma}}{(\beta+\gamma)!} c_{\beta+\gamma} (-\beta-\gamma)_\beta \\ &= (v)_\beta \sum_{|\gamma|=k} (v+\beta)_\gamma c_{\beta+\gamma} \frac{(-\beta-\gamma)_\beta}{(\beta+\gamma)!} = (v)_\beta \sum_{|\gamma|=k} (v+\beta)_\gamma c_{\beta+\gamma} \frac{(-1)^{|\beta|}}{\gamma!} \\ &= (v)_\beta \frac{(-1)^{s-1}}{k!} (s)_k \sum_{|\gamma|=k} \frac{(v+\beta)_\gamma}{(|\beta|+1)_k} c_{\beta+\gamma} \frac{k!}{\gamma!} = (v)_\beta (-1)^{s-1} \binom{n}{s-1} ((R_v^*)^k c)_\beta,\end{aligned}$$

which is zero for all $|\beta| = s-1$ if and only if $(R_v^*)^k c = 0$. □

For Legendre polynomials ($v_j = 1$) and $s = n$ this result appears as Lemma 7 in [10].

The association of the (possibly degree raised) Bernstein coefficients of $f \in \mathcal{P}_s^v$ with a Hahn polynomial preserves the respective inner products.

Theorem 2.6. *Let $f=\sum_{|\alpha|=n} c_\alpha(f)B_\alpha$, $g=\sum_{|\alpha|=n} c_\alpha(g)B_\alpha$ and $0\leqslant s\leqslant n$. If f or g belongs to $\mathcal{P}_s^\nu$, then we have*

$$\langle f,g\rangle_\nu=\frac{(n!)^2}{(n-s)!(|\nu|)_{n+s}}\sum_{|\alpha|=n}\frac{(\nu)_\alpha}{\alpha!}c_\alpha(f)c_\alpha(g)=\frac{(n!)^2}{(n-s)!(|\nu|)_{n+s}}\langle c(f),c(g)\rangle_{\nu,n}.$$

Proof. We will use the multivariate Chu–Vandermonde identity (see [7, §1.2]) in the form

$$\frac{(\nu)_{\alpha+\beta}}{(\nu)_\alpha(\nu)_\beta}=\frac{(\nu+\beta)_\alpha}{(\nu)_\alpha}={}_2F_1(-\alpha,-\beta;\nu;1)=\sum_\gamma\frac{(-\alpha)_\gamma(-\beta)_\gamma}{(\nu)_\gamma\gamma!}.$$

Suppose, without loss of generality that $f\in\mathcal{P}_s^\nu$, and write

$$f=\sum_{|\alpha|=s}a_\alpha B_\alpha,\quad g=\sum_{|\beta|=n}b_\beta B_\beta,$$

i.e., $c(f)=R^{n-s}a$, $c(g)=b$. Then, by (1.2) and Chu–Vandermonde

$$\begin{aligned}\langle f,g\rangle_\nu&=s!n!\sum_{|\alpha|=s}\sum_{|\beta|=n}\frac{a_\alpha}{\alpha!}\frac{b_\beta}{\beta!}\frac{(\nu)_{\alpha+\beta}}{(|\nu|)_{s+n}}\\&=\frac{s!n!}{(|\nu|)_{n+s}}\sum_{|\alpha|=s}\sum_{|\beta|=n}\frac{a_\alpha}{\alpha!}\frac{b_\beta}{\beta!}(\nu)_\alpha(\nu)_\beta\sum_\gamma\frac{(-\alpha)_\gamma(-\beta)_\gamma}{(\nu)_\gamma\gamma!}\\&=\frac{s!n!}{(|\nu|)_{n+s}}\sum_{|\alpha|=s}\frac{(\nu)_\alpha}{\alpha!}a_\alpha\sum_{|\beta|=n}\frac{(\nu)_\beta}{\beta!}b_\beta\sum_{\substack{\gamma\leqslant\alpha\\\gamma\leqslant\beta}}\frac{(-\alpha)_\gamma(-\beta)_\gamma}{(\nu)_\gamma\gamma!}.\end{aligned}$$

Since $p:\alpha\mapsto(-\alpha)_\gamma$ is a polynomial of degree $|\gamma|\leqslant s$ ($\gamma\leqslant\alpha$), Theorem 2.3 (with $n=s$) implies that all terms except those with $\gamma=\alpha$ vanish, and so we obtain

$$\begin{aligned}\langle f,g\rangle_\nu&=\frac{s!n!}{(|\nu|)_{n+s}}\sum_{|\alpha|=s}\frac{(\nu)_\alpha}{\alpha!}a_\alpha\sum_{|\beta|=n}\frac{(\nu)_\beta}{\beta!}b_\beta\frac{(-\alpha)_\alpha(-\beta)_\alpha}{(\nu)_\alpha\alpha!}\\&=\frac{s!n!}{(|\nu|)_{n+s}}\sum_{|\beta|=n}\frac{(\nu)_\beta}{\beta!}b_\beta\sum_{\substack{|\alpha|=s\\\alpha\leqslant\beta}}a_\alpha\frac{(-\beta)_\alpha}{(-\alpha)_\alpha}.\end{aligned}$$

Let $j:=n-s$. Since $(-\beta)_{\beta-\alpha}(-\alpha)_\alpha=(-\beta)_\alpha(-\beta+\alpha)_{\beta-\alpha}$, the last sum above becomes

$$\begin{aligned}\sum_{\substack{|\alpha|=s\\\alpha\leqslant\beta}}a_\alpha\frac{(-\beta)_\alpha}{(-\alpha)_\alpha}&=\sum_{\substack{|\alpha|=s\\\alpha\leqslant\beta}}a_\alpha\frac{(-\beta)_{\beta-\alpha}}{(-(\beta-\alpha))_{\beta-\alpha}}=\sum_{|\gamma|=j}a_{\beta-\gamma}\frac{(-\beta)_\gamma}{(-\gamma)_\gamma}\\&=\frac{(-1)^j}{j!}\sum_{|\gamma|=j}\binom{j}{\gamma}(-\beta)_\gamma a_{\beta-\gamma}=\frac{(-1)^j}{j!}(-n)_j(R^ja)_\beta=\frac{n!}{s!(n-s)!}c_\beta(f),\end{aligned}$$

and we obtain the result. □

For Legendre polynomials ($\nu_j=1$) this is Lemma 6 of [10].

Corollary 2.7. *Let* $f = \sum_{|\alpha|=j} c_\alpha^j(f) B_\alpha$, $g = \sum_{|\alpha|=k} c_\alpha^k(g) B_\alpha$. *If f or g belongs to* $\mathcal{P}_s^\nu$, *where* $s \leqslant \max\{j, k\} \leqslant n$, *then*

$$\langle f, g\rangle_\nu = \frac{(n!)^2}{(n-s)!(|\nu|)_{n+s}} \langle R^{n-j} c^j(f), R^{n-k} c^k(g)\rangle_{\nu,n}.$$

3. Tight frames with symmetries for the Jacobi polynomials

A *tight frame* for a finite dimensional Hilbert space $\mathcal{H}$, such as $\mathcal{P}_s^\nu$, is a sequence of vectors (ϕ_j) in $\mathcal{H}$ for which

$$f = \sum_j \langle f, \phi_j\rangle \phi_j \quad \forall f \in \mathcal{H}. \tag{3.1}$$

Clearly, an orthonormal basis is a tight frame. There do exist tight frames with more vectors than needed for a basis, e.g., three equally spaced vectors in $\mathbb{R}^2$ (see, e.g., [5]).

Proriol's orthonormal basis for $\mathcal{P}_s^\nu$ involved complicated formulae since the symmetries of $\mathcal{P}_s^\nu$ were not utilised. Here, we give a *tight frame* for $\mathcal{P}_s^\nu$ which does have the natural symmetries, and hence has a simple form. Since (3.1) is technically similar to an orthogonal expansion—it simply has more terms—we feel this is a worthwhile advance.

The simple construction (which eluded the author for years) given here is based on the *Bernstein–Durrmeyer* operator M_n^ν (see [6,8]). This is defined on the continuous functions on the simplex T with the Jacobi inner product (1.1) by

$$M_n^\nu f := \sum_{|\alpha|=n} \frac{\langle f, \xi^\alpha\rangle_\nu}{\langle 1, \xi^\alpha\rangle_\nu} B_\alpha = \sum_{|\alpha|=n} \langle f, \xi^\alpha\rangle_\nu \frac{(|\nu|)_n}{(\nu)_\alpha} \frac{n!}{\alpha!} \xi^\alpha.$$

It is easily shown this is self-adjoint and (see [6]) that it has eigenvalues

$$\lambda_s(M_n^\nu) = \frac{n!}{(n-s)!} \frac{1}{(n+|\nu|)_s}, \quad 0 \leqslant s \leqslant n,$$

with corresponding eigenspace the space of Jacobi polynomials $\mathcal{P}_s^\nu$, i.e., for $0 \leqslant s \leqslant n$

$$f = (n-s)!(|\nu|)_{n+s} \sum_{|\alpha|=n} \frac{1}{\alpha!} \frac{\xi^\alpha}{(\nu)_\alpha} \langle f, \xi^\alpha\rangle_\nu \quad \forall f \in \mathcal{P}_s^\nu. \tag{3.2}$$

Let Q_s be the orthogonal projection onto $\mathcal{P}_s^\nu$. Then, for $f \in \mathcal{P}_s^\nu$,

$$\langle f, \xi^\alpha\rangle_\nu = \langle Q_s f, \xi^\alpha\rangle_\nu = \langle f, Q_s(\xi^\alpha)\rangle_\nu,$$

and so from (3.2) we obtain

$$\begin{aligned} f &= (n-s)!(|\nu|)_{n+s} \sum_{|\alpha|=n} \frac{1}{\alpha!} \frac{\xi^\alpha}{(\nu)_\alpha} \langle f, Q_s(\xi^\alpha)\rangle_\nu \\ &= (n-s)!(|\nu|)_{n+s} \sum_{|\alpha|=n} \frac{(\nu)_\alpha}{\alpha!} \frac{Q_s(\xi^\alpha)}{(\nu)_\alpha} \left\langle f, \frac{Q_s(\xi^\alpha)}{(\nu)_\alpha} \right\rangle_\nu. \end{aligned} \tag{3.3}$$

Thus by computing $Q_s(\xi^\alpha)$ explicitly, we obtain the desired tight frame for $\mathcal{P}_s^\nu$.

Lemma 3.4. *Suppose that $|\alpha| = n$. Then for any index γ,*

$$\sum_{|\theta|=s} \frac{(-\alpha)_\theta(-\theta)_\gamma}{\theta!} = (-1)^s \frac{n!}{s!(n-s)!}(-\alpha)_\gamma \frac{(-s)_{|\gamma|}}{(-n)_{|\gamma|}}.$$

Proof. The terms in the sum are nonzero only if $\gamma \leqslant \theta \leqslant \alpha$. Hence, the result holds for $\gamma \nleqslant \alpha$, and it suffices to prove it for the case $\gamma \leqslant \alpha$.

Suppose that $\gamma \leqslant \alpha$. For $\gamma \leqslant \theta \leqslant \alpha$,

$$(-\theta)_\gamma = (-1)^{|\gamma|}\theta!/(\theta-\gamma)!, \quad (-\alpha)_\theta = (-\alpha)_\gamma(-\alpha+\gamma)_{\theta-\gamma},$$

and so, by the multinomial theorem, we have

$$\begin{aligned}\sum_{|\theta|=s} \frac{(-\alpha)_\theta(-\theta)_\gamma}{\theta!} &= \sum_{\substack{|\theta|=s\\ \gamma\leqslant\theta\leqslant\alpha}} \frac{(-\alpha)_\theta(-\theta)_\gamma}{\theta!} = \frac{(-1)^{|\gamma|}(-\alpha)_\gamma}{(s-|\gamma|)!} \sum_{\substack{|\theta|=s\\ \gamma\leqslant\theta\leqslant\alpha}} \frac{(s-|\gamma|)!}{(\theta-\gamma)!}(-\alpha+\gamma)_{\theta-\gamma}\\ &= \frac{(-1)^{|\gamma|}(-\alpha)_\gamma}{(s-|\gamma|)!}(-n+|\gamma|)_{s-|\gamma|}.\end{aligned}$$

Furthermore, since $|\gamma| \leqslant |\theta| = s \leqslant |\alpha| = n$,

$$(s-|\gamma|)! = (-1)^{s-|\gamma|}(-s+|\gamma|)_{s-|\gamma|}, \qquad \frac{(-n+|\gamma|)_{s-|\gamma|}}{(-s+|\gamma|)_{s-|\gamma|}} = \frac{(-n)_s}{(-s)_s}\frac{(-s)_{|\gamma|}}{(-n)_{|\gamma|}},$$

and so we can rearrange this to obtain

$$\sum_{|\theta|=s} \frac{(-\alpha)_\theta(-\theta)_\gamma}{\theta!} = (-1)^s(-\alpha)_\gamma \frac{(-n)_s}{(-s)_s}\frac{(-s)_{|\gamma|}}{(-n)_{|\gamma|}} = (-1)^s \frac{n!}{s!(n-s)!}(-\alpha)_\gamma \frac{(-s)_{|\gamma|}}{(-n)_{|\gamma|}}.$$

This completes the proof. □

Theorem 3.5 (*Tight frame for $\mathcal{P}_s^\nu$*). *Let $n \geqslant s$. A tight frame for $\mathcal{P}_s^\nu$ is given by*

$$f = (n-s)!(|\nu|)_{n+s} \sum_{|\alpha|=n} \frac{(\nu)_\alpha}{\alpha!}\langle f, \phi_\alpha^{\nu,s}\rangle_\nu \phi_\alpha^{\nu,s} \quad \forall f \in \mathcal{P}_s^\nu, \tag{3.6}$$

where

$$\phi_\alpha^{\nu,s} := \frac{(-1)^s}{(s+|\nu|-1)_s}\frac{\binom{n}{s}}{(|\nu|+2s)_{n-s}} \sum_{\substack{\beta\leqslant\alpha\\ |\beta|\leqslant s}} \frac{(s+|\nu|-1)_{|\beta|}(-\alpha)_\beta(-s)_{|\beta|}}{(\nu)_\beta(-n)_{|\beta|}}\frac{\xi^\beta}{\beta!} \tag{3.7}$$

is the orthogonal projection of $\xi^\alpha/(\nu)_\alpha$ onto $\mathcal{P}_s^\nu$. We also have

$$f = \frac{(n-s)!}{n!}(|\nu|)_{n+s} \sum_{|\alpha|=n} \langle f, \phi_\alpha^{\nu,s}\rangle_\nu B_\alpha \quad \forall f \in \mathcal{P}_s^\nu. \tag{3.8}$$

Proof. It suffices to prove $\phi_\alpha^{\nu,s}$ is the orthogonal projection of $\xi^\alpha/(\nu)_\alpha$ onto $\mathcal{P}_s^\nu$, since then (3.3) gives (3.6) and (3.8).

First, we prove this for $|\alpha| = n = s$. In this case

$$\phi_\alpha^v := \phi_\alpha^{v,|\alpha|} := \frac{(-1)^s}{(s+|v|-1)_s} \sum_{\beta\leqslant\alpha} \frac{(s+|v|-1)_{|\beta|}(-\alpha)_\beta}{(v)_\beta} \frac{\xi^\beta}{\beta!} \in \frac{\xi^\alpha}{(v)_\alpha} + \Pi_{s-1}(\mathbb{R}^d). \quad (3.9)$$

For $|\gamma| = s-1$, $(|v|)_{|\beta|+|\gamma|} = (|v|)_{s-1}(|v|+s-1)_{|\beta|}$ $(v)_{\beta+\gamma} = (v)_\gamma(v+\gamma)_\beta$, and so

$$\begin{aligned}\langle \phi_\alpha^v, \xi^\gamma\rangle_v &= \frac{(-1)^s}{(s+|v|-1)_s} \sum_{\beta\leqslant\alpha} \frac{(s+|v|-1)_{|\beta|}(-\alpha)_\beta}{(v)_\beta\beta!} \frac{(v)_{\beta+\gamma}}{(|v|)_{|\beta|+|\gamma|}} \\ &= \frac{(-1)^s}{(s+|v|-1)_s} \frac{(v)_\gamma}{(|v|)_{s-1}} \sum_{\beta\leqslant\alpha} \frac{(-\alpha)_\beta}{(v)_\beta\beta!}(v+\gamma)_\beta.\end{aligned}$$

By Chu–Vandermonde the last sum above equals $(-\gamma)_\alpha/(v)_\alpha = 0$, so ϕ_α^v is orthogonal to the basis $\{\xi^\gamma : |\gamma| = s-1\}$ for $\Pi_{s-1}(\mathbb{R}^d)$, and hence is the orthogonal projection supposed.

Now, we prove the result for $|\alpha| = n \geqslant s$. By (3.8), for $n = s$ (the case just proved), it follows that Q_s the orthogonal projection onto $\mathcal{P}_s^v$ is given by

$$Q_s f = \frac{(|v|)_{2s}}{s!} \sum_{|\beta|=s} \langle Q_s f, \phi_\beta^v\rangle_v B_\beta = \frac{(|v|)_{2s}}{s!} \sum_{|\beta|=s} \langle f, \phi_\beta^v\rangle_v B_\beta \quad \forall f.$$

In particular,

$$\frac{Q_s(\xi^\alpha)}{(v)_\alpha} = \frac{(|v|)_{2s}}{s!} \sum_{|\beta|=s} \left\langle \frac{\xi^\alpha}{(v)_\alpha}, \phi_\beta^v \right\rangle_v B_\beta = \frac{(|v|)_{2s}}{s!} \frac{1}{(v)_\alpha} \sum_{|\beta|=s} \langle \xi^\alpha, \phi_\beta^v\rangle_v B_\beta. \quad (3.10)$$

Let $f = \xi^\alpha = \frac{\alpha!}{n!}B_\alpha$, $|\alpha| = n$, $g = \phi_\beta^v = \sum_{|\theta|=s} c_\theta B_\theta$, $|\beta| = s$ in Corollary 2.7, to obtain:

$$\begin{aligned}\langle \xi^\alpha, \phi_\beta^v\rangle_v &= \frac{(n!)^2}{(n-s)!(|v|)_{n+s}} \left\langle \frac{\alpha!}{n!}\delta_\alpha, R^{n-s}c \right\rangle_{v,n} \\ &= \frac{(n!)^2}{(n-s)!(|v|)_{n+s}} \frac{\alpha!}{n!} \langle (R_v^*)^{n-s}\delta_\alpha, c\rangle_{v,s}. \quad (3.11)\end{aligned}$$

We, therefore, need to calculate $((R_v^*)^{n-s}\delta_\alpha)_\theta$ and c_θ, $|\theta| = s$. By (2.4), we have

$$((R_v^*)^j\delta_\alpha)_\beta = \begin{cases} \frac{(\beta+v)_{\alpha-\beta}}{(|\beta|+1)_j} \frac{j!}{(\alpha-\beta)!}, & \beta\leqslant\alpha; \\ 0 & \text{otherwise} \end{cases} = \frac{(v)_\alpha}{(v)_\beta} \frac{j!}{(|\beta|+1)_j} \frac{(-1)^{|\beta|}(-\alpha)_\beta}{\alpha!},$$

so that

$$((R_v^*)^{n-s}\delta_\alpha)_\theta = \frac{(v)_\alpha}{(v)_\theta} \frac{(n-s)!s!}{n!} \frac{(-1)^s(-\alpha)_\theta}{\alpha!}, \quad |\theta| = s. \quad (3.12)$$

To obtain the Bernstein form $\phi_\alpha^{v,s} = \sum_{|\beta|=m} c_{\alpha,\beta}^{v,s} B_\beta$, $m \geqslant s$, use the multinomial theorem to expand (3.7):

$$\begin{aligned}\phi_\alpha^{v,s} = &\frac{(-1)^s}{(s+|v|-1)_s} \frac{\binom{n}{s}}{(|v|+2s)_{n-s}} \sum_{\substack{\gamma\leqslant\alpha \\ |\gamma|\leqslant s}} \frac{(s+|v|-1)_{|\gamma|}(-\alpha)_\gamma(-s)_{|\gamma|}}{(v)_\gamma(-n)_{|\gamma|}} \frac{\xi^\gamma}{\gamma!} \\ &\times \sum_{|\delta|=m-|\gamma|} \frac{(m-|\gamma|)!}{\delta!}\xi^\delta.\end{aligned}$$

The terms in ξ^β, $|\beta| = m$ $(\beta = \gamma + \delta)$ sum to

$$\frac{(-1)^s}{(s+|v|-1)_s}\,\frac{\binom{n}{s}}{(|v|+2s)_{n-s}}\sum_{\substack{\gamma\leqslant\alpha,\beta\\|\gamma|\leqslant s}}\frac{(s+|v|-1)_{|\gamma|}(-\alpha)_\gamma(-s)_{|\gamma|}}{(v)_\gamma(-n)_{|\gamma|}}\,\frac{\xi^\gamma}{\gamma!}\,\frac{(m-|\gamma|)!}{(\beta-\gamma)!}\xi^{\beta-\gamma}.$$

Since

$$\frac{1}{(\beta-\gamma)!} = (-1)^{|\gamma|}\frac{(-\beta)_\gamma}{\beta!},\quad \gamma\leqslant\beta,\qquad (m-|\gamma|)! = (-1)^{|\gamma|}\frac{m!}{(-m)_{|\gamma|}},$$

this becomes

$$\frac{(-1)^s}{(s+|v|-1)_s}\,\frac{\binom{n}{s}}{(|v|+2s)_{n-s}}\sum_{\substack{\gamma\leqslant\alpha,\beta\\|\gamma|\leqslant s}}\frac{(s+|v|-1)_{|\gamma|}(-\alpha)_\gamma(-\beta)_\gamma(-s)_{|\gamma|}}{(v)_\gamma(-n)_{|\gamma|}(-m)_{|\gamma|}\gamma!}\,\frac{m!}{\beta!}\xi^\beta,$$

i.e.,

$$c^{v,s}_{\alpha,\beta} = \frac{(-1)^s}{(s+|v|-1)_s}\,\frac{\binom{|\alpha|}{s}}{(|v|+2s)_{|\alpha|-s}}\sum_{\substack{\gamma\leqslant\alpha,\beta\\|\gamma|\leqslant s}}\frac{(s+|v|-1)_{|\gamma|}(-\alpha)_\gamma(-\beta)_\gamma(-s)_{|\gamma|}}{(v)_\gamma(-|\alpha|)_{|\gamma|}(-|\beta|)_{|\gamma|}\gamma!}. \tag{3.13}$$

As a particular case, we have

$$c_\theta = c^{v,s}_{\beta,\theta} = \frac{(-1)^s}{(s+|v|-1)_s}\sum_{\substack{\gamma\leqslant\beta,\theta\\|\gamma|\leqslant s}}\frac{(s+|v|-1)_{|\gamma|}(-\beta)_\gamma(-\theta)_\gamma}{(v)_\gamma(-s)_{|\gamma|}\gamma!},\quad |\theta| = s. \tag{3.14}$$

Combining (3.11), (3.12) and (3.14), we obtain

$$\begin{aligned}\langle\xi^\alpha,\phi^v_\beta\rangle_v &= \frac{s!}{(|v|)_{n+s}}\,\frac{(v)_\alpha}{(s+|v|-1)_s}\sum_{|\theta|=s}\frac{(-\alpha)_\theta}{\theta!}\sum_{\substack{\gamma\leqslant\beta\\|\gamma|\leqslant s}}\frac{(s+|v|-1)_{|\gamma|}(-\beta)_\gamma(-\theta)_\gamma}{(v)_\gamma(-s)_{|\gamma|}\gamma!}\\ &= \frac{s!}{(|v|)_{n+s}}\,\frac{(v)_\alpha}{(s+|v|-1)_s}\sum_{\substack{\gamma\leqslant\beta\\|\gamma|\leqslant s}}\frac{(s+|v|-1)_{|\gamma|}(-\beta)_\gamma}{(v)_\gamma(-s)_{|\gamma|}\gamma!}\sum_{|\theta|=s}\frac{(-\alpha)_\theta}{\theta!}(-\theta)_\gamma.\end{aligned} \tag{3.15}$$

By Lemma 3.4,

$$\sum_{|\theta|=s}\frac{(-\alpha)_\theta(-\theta)_\gamma}{\theta!} = \sum_{\substack{|\theta|=s\\\gamma\leqslant\theta\leqslant\alpha}}\frac{(-\alpha)_\theta(-\theta)_\gamma}{\theta!} = (-1)^s\binom{n}{s}(-\alpha)_\gamma\frac{(-s)_{|\gamma|}}{(-n)_{|\gamma|}}. \tag{3.16}$$

So by (3.10), (3.15) and (3.16), the B_β–Bernstein coefficient ($|\beta| = s$) of the orthogonal projection of $\xi^\alpha/(v)_\alpha$ onto $\mathcal{P}^v_s$ is

$$\frac{(|v|)_{2s}}{s!}\,\frac{1}{(v)_\alpha}\langle\xi^\alpha,\phi^v_\beta\rangle_v = \frac{(-1)^s}{(s+|v|-1)_s}\,\frac{(|v|)_{2s}}{(|v|)_{n+s}}\binom{n}{s}\sum_{\substack{\gamma\leqslant\alpha,\beta\\|\gamma|\leqslant s}}\frac{(s+|v|-1)_{|\gamma|}(-\alpha)_\gamma(-\beta)_\gamma}{(v)_\gamma(-n)_{|\gamma|}\gamma!}.$$

This equals the corresponding Bernstein coefficient $c^{\nu,s}_{\alpha,\beta}$ given by (3.13), and so $\phi^{\nu,s}_\alpha$ is the orthogonal projection of $\xi^\alpha/(\nu)_\alpha$ onto $\mathcal{P}^\nu_s$ as claimed. $\square$

The tight frame (3.6) and the representation (3.8) have the desired symmetries. For applications, one would use the tight frame with the smallest number of vectors, i.e., that for $n = s$, which simplifies to

$$f = (|\nu|)_{2s} \sum_{|\alpha|=s} \frac{(\nu)_\alpha}{\alpha!} \langle f, \phi^\nu_\alpha \rangle_\nu \phi^\nu_\alpha = \frac{(|\nu|)_{2s}}{s!} \sum_{|\alpha|=s} \langle f, \phi^\nu_\alpha \rangle_\nu B_\alpha \quad \forall f \in \mathcal{P}^\nu_s,$$

where

$$\phi^\nu_\alpha := \phi^{\nu,|\alpha|}_\alpha = \frac{(-1)^s}{(s+|\nu|-1)_s} \sum_{\substack{\beta\leqslant\alpha \\ |\beta|\leqslant s}} \frac{(s+|\nu|-1)_{|\beta|}(-\alpha)_\beta}{(\nu)_\beta} \frac{\xi^\beta}{\beta!}, \quad |\alpha| = s. \tag{3.17}$$

For example, in the univariate case, with $\xi(x) = (1-x, x)$ the barycentric coordinates for the interval $[0, 1]$ and $\alpha = (j, s-j)$, $0 \leqslant j \leqslant s$, (3.17) becomes

$$\phi^{(\nu_0,\nu_1)}_{(j,s-j)}(x) = \frac{(-1)^s}{(s+\nu_0+\nu_1-1)_s} \sum_{\substack{\beta_0\leqslant j \\ \beta_1\leqslant s-j}} \frac{(s+\nu_0+\nu_1-1)_{\beta_0+\beta_1}(-j)_{\beta_0}(j-s)_{\beta_1}}{(\nu_0)_{\beta_0}(\nu_1)_{\beta_1}} \times \frac{(1-x)^{\beta_0}x^{\beta_1}}{\beta_0!\beta_1!}.$$

This was (with hindsight) easily proved, and can also be proved using [15, Corollary 5.8].

The sum in (3.17) is a multivariate generalisation of the ${}_2F_1$ hypergeometric function called the Lauricella function of type A (see [7, §1.2]). In the univariate case (3.17) reduces to the usual formula for Jacobi polynomials. The history of Theorem 3.5 is as follows. The case $n = s$ was proved in [14] (unpublished) and is closely connected with results in [12] (see [13]). The simple proof is new, as are the formulas involving the Bernstein form.

From (3.13) in the proof we have the following.

Corollary 3.18 (*Bernstein form of* $\phi^{\nu,s}_\alpha$). *The projection of* $\xi^\alpha/(\nu)_\alpha$, $|\alpha| = n$ *onto* $\mathcal{P}^\nu_s$ *can be written*

$$\phi^{\nu,s}_\alpha = \sum_{|\beta|=m} c^{\nu,s}_{\alpha,\beta} B_\beta, \quad s \leqslant m \leqslant n,$$

where

$$c^{\nu,s}_{\alpha,\beta} = \frac{(-1)^s}{(s+|\nu|-1)_s} \frac{\binom{n}{s}}{(|\nu|+2s)_{n-s}} \sum_{\substack{\gamma\leqslant\alpha,\beta \\ |\gamma|\leqslant s}} \frac{(s+|\nu|-1)_{|\gamma|}(-\alpha)_\gamma(-\beta)_\gamma(-s)_{|\gamma|}}{(\nu)_\gamma(-n)_{|\gamma|}(-m)_{|\gamma|}\gamma!}.$$

Since $\Pi_n(\mathbb{R}^d)$ is the orthogonal direct sum $\oplus^n_{s=0}\mathcal{P}^\nu_s$, we obtain the following tight frame.

Corollary 3.19 (*Tight frame for* Π_n). *A tight frame for* $\Pi_n(\mathbb{R}^d)$ *with* $\langle\cdot,\cdot\rangle_\nu$ *is given by*

$$f = \sum_{s=0}^{n} (|\nu|)_{2s} \sum_{|\alpha|=s} \frac{(\nu)_\alpha}{\alpha!} \langle f, \phi_\alpha^\nu\rangle_\nu \phi_\alpha^\nu \in \bigoplus_{s=0}^{n} \mathcal{P}_s^\nu \quad \forall f \in \Pi_n(\mathbb{R}^d), \tag{3.20}$$

where $\langle \phi_\alpha^\nu, \phi_\beta^\nu\rangle_\nu = 0$, $|\alpha| \neq |\beta|$.

Theorem 3.21 (*Projection matrix*). *The matrix* $A = (a_{\alpha\beta})$ *that maps the Bernstein coefficients of* $f = \sum_{|\beta|=n} c_\beta B_\beta$ *to those of* $Q_s f = \sum_{|\alpha|=m} (Ac)_\alpha B_\alpha$ *its projection onto* $\mathcal{P}_s^\nu$ *is given by*

$$a_{\alpha\beta} = n! \frac{(\nu)_\beta}{\beta!} c_{\beta,\alpha}^{\nu,s}, \quad |\alpha| = m \geqslant s, \quad |\beta| = n.$$

Proof. Since $Q_s(B_\beta) = (n!/\beta!)(\nu)_\beta \phi_\beta^{\nu,s}$, Corollary 3.18 gives

$$\begin{aligned} Q_s f &= n! \sum_{|\beta|=n} \frac{(\nu)_\beta}{\beta!} c_\beta \phi_\beta^{\nu,s} = n! \sum_{|\beta|=n} \frac{(\nu)_\beta}{\beta!} c_\beta \sum_{|\alpha|=m} c_{\beta,\alpha}^{\nu,s} B_\alpha \\ &= \sum_{|\alpha|=m} \left(\sum_{|\beta|=n} n! \frac{(\nu)_\beta}{\beta!} c_{\beta,\alpha}^{\nu,s} c_\beta \right) B_\alpha. \qquad \square \end{aligned} \tag{3.22}$$

For $m = s$, the entries of the matrix A are given by a multivariate analogue of the ${}_3F_2$ hypergeometric function.

Since $(\phi_\alpha^{\nu,s})_{|\alpha|=n}$ spans $\mathcal{P}_s^\nu$, the orthogonal projection of $f = \sum_{|\beta|=n} c_\beta B_\beta$ onto $\mathcal{P}_s^\nu$ can be expressed as follows:

$$Q_s f = \sum_{|\alpha|=n} \frac{(\nu)_\alpha}{\alpha!} b_\alpha \phi_\alpha^{\nu,s},$$

where by (3.6) and a result from the theory of frames, the unique coefficients $b = (b_\alpha)$ with the minimal ℓ_2-norm are given by

$$b_\alpha = (n-s)!(|\nu|)_{n+s} \langle f, \phi_\alpha^{\nu,s}\rangle_\nu = (n-s)!(|\nu|)_{n+s} \sum_{|\beta|=n} \langle B_\beta, \phi_\alpha^{\nu,s}\rangle_\nu c_\beta.$$

There is a simpler choice for the coefficients b_α which gives the projection.

Corollary 3.23. *The orthogonal projection of* $f = \sum_{|\alpha|=n} c_\alpha B_\alpha \in \Pi_n(\mathbb{R}^d)$ *onto* $\mathcal{P}_s^\nu$, $0 \leqslant s \leqslant n$, *is given by*

$$Q_s f = n! \sum_{|\alpha|=n} \frac{(\nu)_\alpha}{\alpha!} c_\alpha \phi_\alpha^{\nu,s}. \tag{3.24}$$

Proof. Apply Q_s to $f = \sum_{|\alpha|=n} c_\alpha B_\alpha$. $\square$

Since $\langle B_\beta, \phi_\alpha^{\nu,s}\rangle_\nu \neq 0$, $|\alpha| = |\beta| = n$, (3.24) is not the tight frame representation of (3.6).

4. Applications and further results

Here, we apply our results to a surface "smoothing" problem considered in [10] (where the Bernstein form of the Prorial basis was used). We adapt the notation used there.Consider a triangular surface patch of total degree n, expressed both in the Bernstein basis and the tight frame (3.20),

$$f = \sum_{|\alpha|=n} p_\alpha B_\alpha = \sum_{s=0}^{n} (|\nu|)_{2s} \sum_{|\beta|=s} \frac{(\nu)_\beta}{\beta!} \langle f, \phi_\beta^\nu \rangle_\nu \phi_\beta^\nu = \sum_{s=0}^{n} \sum_{|\beta|=s} q_\beta^s \phi_\beta^\nu.$$

Using Corollary 3.18, the coefficients (p_α) can be computed from (q_β^s). But since (ϕ_β^s) is not a basis there is not a unique choice for a given f. To get a uniqueness, we require that q_β^s be the coefficients given by (3.20), i.e., $(|\nu|)_{2s}(\nu)_\beta \langle f, \phi_\beta^\nu \rangle_\nu / \beta! = q_\beta^s$, which simplifies to

$$(|\nu|)_{2s} \frac{(\nu)_\beta}{\beta!} \sum_{|\alpha|=s} \langle \phi_\alpha^\nu, \phi_\beta^\nu \rangle_\nu q_\alpha^j = q_\beta^s, \quad |\beta| = s, \;\; 0 \leqslant s \leqslant n. \tag{4.1}$$

Notice that these equations have dependencies (which allows a system with the natural symmetries). Assume M linear interpolation conditions have been prescribed

$$\lambda_j(f) = \sum_{s=0}^{n} \sum_{|\beta|=s} \lambda_j(\phi_\beta^n) q_\beta^s = a_j, \quad 1 \leqslant j \leqslant M, \tag{4.2}$$

e.g., specifying the boundary curves. Then the remaining degrees of freedom are used to minimise a "surface smoothness" integral

$$J(q_\beta^j : |\beta| = j, 0 \leqslant j \leqslant s) := \int_T \Psi(f) \xi^{\nu-1},$$

where $\Psi(f)$ is quadratic in f and its derivatives. The orthogonality $\langle \phi_\alpha^\nu, \phi_\beta^\nu \rangle_\nu$, $|\alpha| \neq |\beta|$ ensures that the system of equations obtained from the method of Lagrange multipliers takes a simple form (cf. [10]). The resulting "smoothest" surface obtained is the same as that in [10], with our calculation treating all vertices equally.

It follows from Corollary 2.7 that repeated applications of R and R_ν^* to the Bernstein coefficients of a Jacobi polynomial take a simple form.

Theorem 4.3. *Let* $f = \sum_{|\alpha|=n} c_\alpha B_\alpha \in \mathcal{P}_s^\nu$, $n \geqslant s$. *For* $j, k \geqslant 0$ *with* $n + k - j \geqslant 0$,

$$(R_\nu^*)^j R^k c = \frac{(n-s+k-j+1)_j}{(n+k-j+1)_j^2} (|\nu| + n + s + k - j)_j R^{k-j} c, \quad j \leqslant k,$$

$$(R_\nu^*)^j R^k c = \frac{(n-s+1)_k}{(n+1)_k^2} (|\nu| + n + s)_k (R_\nu^*)^{j-k} c, \quad k \leqslant j.$$

Note by Corollary 2.5, $(R_\nu^*)^j R^k c = 0$ *if* $j - k > n - s$, *i.e.*, $n + k - j > s$.

Proof. Let $g = \sum_{|\beta|=n+k-j} b_\beta B_\beta$. First, suppose $j \leqslant k$. Then, by two applications of Corollary 2.7, we have:

$$\begin{aligned}\langle f, g\rangle_v &= \frac{((n+k-j)!)^2}{(n+k-j-s)!(|v|)_{n+k-j+s}} \langle R^{k-j} c, b\rangle_{v,n+k-j} \\ &= \frac{((n+k)!)^2}{(n+k-s)!(|v|)_{n+k+s}} \langle R^k c, R^j b\rangle_{v,n+k} \\ &= \frac{((n+k)!)^2}{(n+k-s)!(|v|)_{n+k+s}} \langle (R_v^*)^j R^k c, b\rangle_{v,n+k-j}.\end{aligned}$$

Since b is arbitrary, the first arguments in the inner products $\langle \cdot, b\rangle_{n+k-j}$ are equal, giving

$$\frac{((n+k)!)^2}{(n+k-s)!(|v|)_{n+k+s}} (R_v^*)^j R^k c = \frac{((n+k-j)!)^2}{(n+k-j-s)!(|v|)_{n+k-j+s}} R^{k-j} c$$

as supposed. Similarly, for $k \leqslant j$, use

$$\begin{aligned}\langle f, g\rangle_v &= \frac{(n!)^2}{(n-s)!(|v|)_{n+s}} \langle c, R^{j-k} b\rangle_{v,n} = \frac{(n!)^2}{(n-s)!(|v|)_{n+s}} \langle (R_v^*)^{j-k} c, b\rangle_{v,n+k-j} \\ &= \frac{((n+k)!)^2}{(n+k-s)!(|v|)_{n+k+s}} \langle R^k c, R^j b\rangle_{v,n+k} \\ &= \frac{((n+k)!)^2}{(n+k-s)!(|v|)_{n+k+s}} \langle (R_v^*)^j R^k c, b\rangle_{v,n+k-j}. \qquad \square\end{aligned}$$

For Legendre polynomials ($v = 1$) and $j \leqslant k$ this appears as Lemma 8 in [10].

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